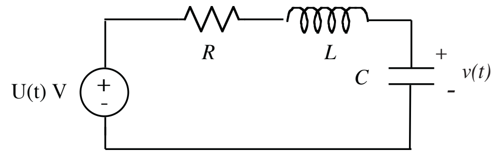
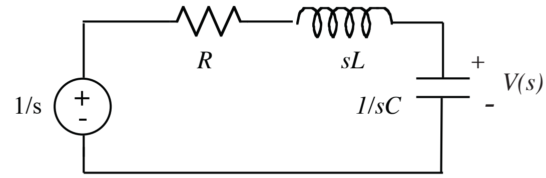
2^{nd} Order Step Response

For the circuit:





Using the voltage divider: $V(s) = \dfrac{1/LC}{s\left(s^2 + \dfrac{R}{L}s + \dfrac{1}{LC}\right)}$. The roots of quadratic term are:

$s_{1,2} = -\dfrac{R}{2L} \pm \sqrt{\left(\dfrac{R}{2L}\right)^2 - \dfrac{1}{LC}}$, which can be expressed as: $s_{1,2} = \omega_0\left[-\varsigma \pm j\sqrt{1-\varsigma^2}\right]$, where

$\varsigma = \dfrac{R}{2}\sqrt{\dfrac{C}{L}}$ is the *damping factor* and $\omega_0 = \dfrac{1}{\sqrt{LC}}$ is the *resonant frequency*. Here it is clear that the damping factor controls whether the response will be underdamped ($0 < \varsigma < 1$), critically damped ($\varsigma = 1$), or overdamped ($\varsigma > 1$). We can also express the roots as: $s_{1,2} = -\alpha \mp j\beta$, where $\alpha = \omega_0\varsigma$ is the *attenuation factor*, and $\beta = \omega_0\sqrt{1-\varsigma^2}$ is the *ring frequency* (these names will be clear below). Using these definitions, $V(s)$ can be written as:

$$V(s) = \frac{\omega_0^2}{s\left(s^2 + 2\varsigma\omega_0 s + \omega_0^2\right)} = \frac{\omega_0^2}{s(s+\alpha-j\beta)(s+\alpha+j\beta)} = \frac{\omega_0^2}{s\left((s+\alpha)^2+\beta^2\right)}.$$

Using partial fraction expansion, this can be written as:

$$V(s) = \frac{\omega_0^2}{s\left((s+\alpha)^2+\beta^2\right)} = \frac{A}{s} + \frac{D(s+\alpha)}{(s+\alpha)^2+\beta^2} + \frac{E\beta}{(s+\alpha)^2+\beta^2}.$$

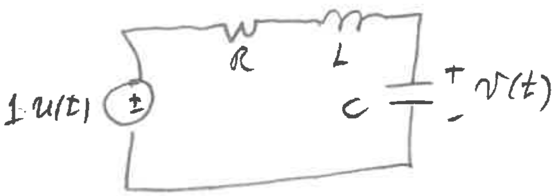
Solving for the unknown constants, we obtain: $A = -D = 1$ and $E = -\dfrac{\alpha}{\beta} = -\dfrac{\varsigma}{\sqrt{1-\varsigma^2}}$, so

we have: $V(s) = \dfrac{1}{s} - \dfrac{(s+\alpha)}{(s+\alpha)^2+\beta^2} - \dfrac{\varsigma}{\sqrt{1-\varsigma^2}}\dfrac{\beta}{(s+\alpha)^2+\beta^2}$, which transforms back to

the time-domain as:

$$v(t) = \left[1 - e^{-\alpha t}\cos\beta t - \frac{\varsigma}{\sqrt{1-\varsigma^2}}e^{-\alpha t}\sin\beta t\right]U(t) = \left[1 - \frac{1}{\sqrt{1-\varsigma^2}}e^{-\alpha t}\cos\left(\beta t - \tan^{-1}\left(\frac{\varsigma}{\sqrt{1-\varsigma^2}}\right)\right)\right]U(t)$$

2nd Order Step Response



For $L = C = 1$

$\xi = R/2$ (squig $\doteq$ damping factor)

$N := 500 \qquad Tmax := 10 \qquad i := 0, 1 .. N$

$L := 1 \qquad C := 1 \qquad t(i) := Tmax \cdot \dfrac{i}{N} \qquad omeg0 := \dfrac{1}{\sqrt{L \cdot C}}$

$squig(R) := \dfrac{R}{2} \cdot \sqrt{\dfrac{C}{L}} \qquad alpha(R) := omeg0 \cdot squig(R) \qquad beta(R) := omeg0 \cdot \sqrt{1 - squig(R)^2}$

$$v(i, R) := 1 - e^{-alpha(R) \cdot t(i)} \cdot \left(\cos(beta(R) \cdot t(i)) + \frac{squig(R)}{\sqrt{1 - squig(R)^2}} \cdot \sin(beta(R) \cdot t(i)) \right)$$

$$vc(i) := 1 - (1 + omeg0 \cdot t(i)) \cdot e^{-omeg0 \cdot t(i)}$$